USE OF VARIATIONAL METHODS IN CONSTRUCTING MODELS OF CONTINUOUS MEDIA WITH IRREVERSIBLE PROCESSES IN THE SPECIAL THEORY OF RELATIVITY

(VARIATSIONNYE METODY POSTROENIIA MODELEI SPLOSHNYKH SRED S NEOBRATNYMI PROTSESSAMI V SPETSIAL'NOI TEORII OTNOSITEL'NOSTI)

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1. Introduction. Construction of models using a minimum number of hypotheses is one of the major problems of the general theory of models of continuous media. Variational principle

$$\delta \int_V \Lambda \, d\tau + \delta W + \delta W^* = 0 \qquad (1)$$

formulated in [1 and 2], can be utilized for that purpose and construction of the model can be reduced to obtaining the Lagrangian Λ and the functional δW^*.

We shall consider, within the framework of the special theory of relativity continuous media, for which the Lagrangian Λ is dependent on (*)

$$\frac{\partial x^i}{\partial \xi^p} \equiv x^i_{\ p}, \quad \nabla_k x^i_{\ p}, \quad \mu^A, \quad \nabla_j \mu^A, \quad \nabla_i \nabla_j \mu^A, \quad S, \quad K^C, \quad L^{\hat{}C} \qquad (2)$$

Here x^i are the coordinates in the fixed observer system which is, generally, curvilinear, ξ^p are the coordinates in the moving coordinate system, functions $x^i(\xi^p)$ define the transition from the moving to the observer's coordinate system and give the law of motion of the medium, μ^A are the field functions, scalar S is the entropy measured by the observer in his system and referred to the unit rest mass, K^C are the components of some given tensors in the observer's system which are not subject to variation and $L^{\hat{}C}$ are the corresponding magnitudes in the moving coordinate system (**). Components K^C and $L^{\hat{}C}$ represent here either physical constants, or their generalizations. Covariant derivative ∇_j is, by definition, taken in the observer's system.

Dependence of Λ on the arguments $x^i, x^i_{\ p}, \mu^A, \nabla_j \mu^A$ and K^C was discussed by Sedov in [1]. The same problem with $\delta W^* = 0$ was treated within the framework of the general theory of relativity in [3]. In the present paper we study the relationships which may be of use in determining the functional δW^*.

*) Indices $i, j, k, \ldots$ assume the values 1, 2, 3 and 4. Indices A, B and C may correspond to one or several tensor indices. Indices $\alpha, \beta, \gamma, \ldots$ assume the values 1, 2, 3 and 4 and correspond to spatial coordinates.

**) Here and in the following $\hat{}$ denotes the moving coordinate system.

2. Basic equations. By definition, we shall assume (*)

$$\delta W^* = \int\limits_V [\rho\Theta\delta S - Q_i\delta x^i - Q_A\delta\mu^A]\,d\tau -$$

$$- \int\limits_\Sigma [Q_i{}^l\delta x^i + Q_i{}^{jl}\delta x^i{}_j + Q_A{}^l\,\delta\mu^A + Q_A{}^{jl}\delta(\nabla_j\mu^A)]\,n_l\,d\sigma \qquad (3)$$

Here the scalar ρ is the density of the rest mass, while Θ, Q_i, Q_A, $Q_i{}^l$, $Q_i{}^{jl}$, $Q_A{}^l$ and $Q_A{}^{jl}$ are some arbitrary functions or functionals.

Performing the variation in (1), we find

$$\delta W = \int\limits_\Sigma \{P_i{}^l\delta x^i + P_i{}^{jl}\delta x^i{}_j + P_A{}^l\delta\mu^A + P_A{}^{jl}\delta(\nabla_j\mu^A)\}\,n_l\,d\sigma$$

where

$$P_i{}^j = \nabla_i\mu^A \frac{\partial\Lambda}{\partial\nabla_j\mu^A} + \frac{\partial\Lambda}{\partial\nabla_s\nabla_k\mu^A}(\delta_k{}^j\nabla_s\nabla_i\mu^A + \delta_s{}^j\nabla_k\nabla_i\mu^A) -$$

$$- x^j{}_s\frac{\partial\Lambda}{\partial\nabla_k x^l{}_s}(\delta_i{}^l\nabla_k x^j{}_s - \delta_k{}^j\nabla_i x_s{}^l) - \nabla_l\left\{\nabla_i\mu^A\frac{\partial\Lambda}{\partial\nabla_j\nabla_l\mu^A} - \frac{1}{2}\left[x^l{}_s\frac{\partial\Lambda}{\partial\nabla_j x_s{}^i} + \right.\right.$$

$$\left.\left. + x^j{}_s\frac{\partial\Lambda}{\partial\nabla_l x^i{}_s}\right]\right\} - \Lambda\delta_i{}^j + Q_i{}^j = P_{(\Lambda)i}{}^j + Q_i{}^j$$

$$P_i{}^{jl} = -\frac{1}{2}\left[x^l{}_s\frac{\partial\Lambda}{\partial\nabla_k x^i{}_s} + x^k{}_s\frac{\partial\Lambda}{\partial\nabla_l x^i{}_s}\right]\frac{\partial\xi^j}{\partial x^k} + Q_i{}^{jl} = P_{(\Lambda)i}{}^{jl} + Q_i{}^{jl}$$

$$P_A{}^j = -\frac{\partial\Lambda}{\partial\nabla_j\mu^A} + \nabla_l\frac{\partial\Lambda}{\partial\nabla_j\nabla_l\mu^A} + Q_A{}^j = P_{(\Lambda)A}{}^j + Q_A{}^j$$

$$P_A{}^{jl} = \frac{\partial\Lambda}{\partial\nabla_j\nabla_l\mu^A} + Q_A{}^{jl} = P_{(\Lambda)A}{}^{jl} + Q_A{}^{jl} \qquad (4)$$

$P_{(\Lambda)i}{}^j$, $P_{(\Lambda)i}{}^{jl}$, $P_{(\Lambda)A}{}^j$ and $P_{(\Lambda)A}{}^{jl}$ denote parts of the tensors $P_i{}^j$, $P_i{}^{jl}$, $P_A{}^j$ and $P_A{}^{jl}$, which can be determined provided the Lagrangian Λ is given.

Tensor $P_i{}^j$ shall, after its determination, be called the energy impulse tensor.

Variational principle (1) also yields the system

$$\nabla_j P_{(\Lambda)i}{}^j = Q_i \qquad (5)$$

$$\frac{\partial\Lambda}{\partial\mu^A} - \nabla_j\frac{\partial\Lambda}{\partial\nabla_j\mu^A} + \nabla_i\nabla_j\frac{\partial\Lambda}{\partial\nabla_i\nabla_j\mu^A} = Q_A \qquad (6)$$

$$\frac{\partial\Lambda}{\partial S} = -\rho\Theta \qquad (7)$$

which, together with equations of state (4), fully describe the behavior of the continuous medium. In particular, (5) to (7) contain the equation of entropy, which can be written [5 to 7] as follows:

$$\rho DS = \nabla_k H^k + \sigma,\ D = u^k\nabla_k \qquad (8)$$

where u^k is the 4-velocity vector, while the vector H^k together with the scalar σ satisfy

$$H^k u_k = 0, \qquad \sigma \geqslant 0 \qquad (9)$$

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*) Vector δx^i in (3) is a variation of world lines (see [1 to 3] for variations).

Let us find the right-hand side of (8) in the model constructed on the basis of the variational principle. To do this, we shall use equation of the heat influx [1, 4, 5, 6 and 7], which can be written as

$$x_4^i \nabla_i P_{(\Lambda)i}^{\ j} = x_4^{\ i} Q_i \tag{10}$$

Inserting into (10) Expression for $P_{(\Lambda)i}^{\ j}$ from (4) and taking into account

$$x_4^{\ i}\nabla_i\Lambda = \frac{\partial\Lambda}{\partial x^j_{\ s}} x^i_{\ 4}\nabla_i x^j_{\ s} + \frac{\partial\Lambda}{\partial\nabla_k x^l_{\ s}} x^i_{\ 4}\nabla_i\nabla_k x^l_{\ s} + \frac{\partial\Lambda}{\partial\mu^A} x^i_{\ 4}\nabla_i\mu^A + \frac{\partial\Lambda}{\partial\nabla_j\mu^A} x^i_{\ 4}\nabla_i\nabla_j\mu^A +$$

$$+ \frac{\partial\Lambda}{\partial\nabla_j\nabla_k\mu^A} x^i_{\ 4}\nabla_i\nabla_j\nabla_k\mu^A + \frac{\partial\Lambda}{\partial S} x^i_{\ 4}\nabla_i S + \frac{\partial\Lambda}{\partial K^C} x^i_{\ 4}\nabla_i K^C + \frac{\partial\Lambda}{\partial L^{\wedge C}}\frac{\partial L^{\wedge C}}{\partial\xi^4}$$

$$x^j_{\ s}\nabla_j x^i_{\ 4} - x^j_{\ 4}\nabla_j x^i_{\ s} = \frac{\partial x^i_{\ 4}}{\partial\xi^s} - \frac{\partial x^i_{\ s}}{\partial\xi^4} = 0$$

$$x^j_{\ s}\nabla_j\nabla_k x^i_{\ 4} - x^j_{\ 4}\nabla_j\nabla_k x^i_{\ s} + \nabla_j x^l_{\ 4}(\delta_l^{\ i}\nabla_k x^j_{\ s} - \delta_k^{\ j}\nabla_l x^i_{\ s}) = \nabla_k(x^j_{\ s}\nabla_j x^i_{\ 4} - x^j_{\ 4}\nabla_j x^i_{\ s}) = 0$$

we obtain

$$\rho\Theta x^i_{\ 4}\nabla_i S = \nabla_j F^j + \frac{\partial\Lambda}{\partial K^C} x^i_{\ 4}\nabla_i K^C + \frac{\partial\Lambda}{\partial L^{\wedge C}}\frac{\partial L^{\wedge C}}{\partial\xi^4} + Q_i x^i_{\ 4} + Q_A x^i_{\ 4}\nabla_i\mu^A \tag{11}$$

where

$$F^j = \frac{\partial\Lambda}{\partial x^i_{\ s}} x^i_{\ 4} x^j_{\ s} + \frac{\partial\Lambda}{\partial\nabla_k x^l_{\ s}}(x^l_{\ 4}\nabla_k x^j_{\ s} - \delta_k^{\ j} x^i_{\ 4}\nabla_i x^l_{\ s}) - \frac{1}{2} x^i_{\ 4}\nabla_l\left[x_s^{\ l}\frac{\partial\Lambda}{\partial\nabla_j x^i_{\ s}} + \right.$$

$$\left. + x^j_{\ s}\frac{\partial\Lambda}{\partial\nabla_l x^i_{\ s}}\right] + \frac{1}{2}\left[x^l_{\ s}\frac{\partial\Lambda}{\partial\nabla_j x^i_{\ s}} + x^j_{\ s}\frac{\partial\Lambda}{\partial\nabla_l x^i_{\ s}}\right]\nabla_l x^i_{\ 4} \tag{12}$$

Using (*)

$$x^i_{\ 4} = \sqrt{g^{\wedge}_{\ 44}}\cdot u^i, \qquad g^{\wedge}_{\ 44} = g_{lm} x^l_{\ 4} x^m_{\ 4}$$

to replace the $x_4^{\ i}$ vector in (11) with the 4-velocity vector u^i, we obtain in final form of (11)

$$\rho DS = \frac{1}{\Theta}\left[\frac{1}{\sqrt{g^{\wedge}_{\ 44}}}\nabla_j F^j + \frac{\partial\Lambda}{\partial K^C} DK^C + \frac{1}{\sqrt{g^{\wedge}_{\ 44}}}\frac{\partial\Lambda}{\partial L^{\wedge C}}\frac{\partial L^{\wedge C}}{\partial\xi^4} + Q_i u^i + Q_A D\mu^A\right] \tag{13}$$

which shows, that the entropy change is the result of not only the work done by generalized forces Q_i and Q_A, but also of action of energy sources defined by the Lagrangian Λ. From (13) it follows also, that the entropy of a particle will vary, if its physical or geometrical characteristics K^C and $L^{\wedge C}$ vary with time.

Equation (13) can be reduced to (8), if additional hypotheses based on the physical sense of $\mu^A, K^C, L^{\wedge C}, \Theta$ and Λ, are introduced.

3. Models defined by the given Lagrangian Λ. Let us consider the models, for which generalized forces Q_i and Q_A are equal to zero. Equation of entropy balance has, for such models, the form

$$\rho DS = \frac{1}{\Theta}\left[\frac{1}{\sqrt{g^{\wedge}_{\ 44}}}\nabla_j F^j + \frac{\partial\Lambda}{\partial K^C} DK^C + \frac{1}{\sqrt{g^{\wedge}_{\ 44}}}\frac{\partial\Lambda}{\partial L^{\wedge C}}\frac{\partial L^{\wedge C}}{\partial\xi^4}\right] \tag{14}$$

It should be noted, that the vector F^j is independent of field functions and their derivatives, provided that Λ has the form

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*) Here g_{ij} denote components of the metric tensor in the observer's system.

$$\Lambda^{(1)}(x^i{}_j,\ \nabla_k x^i{}_j,\ K^C,\ L^{\wedge C}) + \Lambda^{(2)}(\mu^A,\ \nabla_j\mu^A,\ \nabla_i\nabla_j\mu^A) \quad (14)$$

Here $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are the Lagrangians of matter and field respectively.

In particular, the above situation occurs in case of electromagnetic field in neutral medium. We shall show, that in the number of cases the right-hand side of (14) becomes equal to zero.

1) We have an ideal compressible fluid without the thermal conductivity: $\Lambda = \Lambda(\rho, S)$, while θ is in terms of the absolute temperature T.

By the definition of density ρ [8 and 3], we have

$$\frac{\partial\rho}{\partial x^i{}_s} = \frac{\partial\rho}{\partial\gamma^\wedge{}_{pq}}\frac{\partial\gamma^\wedge{}_{pq}}{\partial x^i{}_s} = -\frac{1}{2}\rho g^{\wedge pq}\frac{\partial\gamma^\wedge{}_{pq}}{\partial x^i{}_s} = -\rho\gamma_i{}^k\frac{\partial\xi^s}{\partial x^k}$$

where $\gamma_{pq}{}^\wedge = g_{pq}{}^\wedge - u_p{}^\wedge u_q{}^\wedge$. Consequently

$$\frac{\partial\rho}{\partial x^i{}_s}x^j{}_s = -\rho\gamma_i{}^j \quad (15)$$

which we use to find

$$F^j = \frac{\partial\Lambda}{\partial\rho}\frac{\partial\rho}{\partial x^i{}_s}x^j{}_s x^i{}_4 = -\rho\,\frac{\partial\Lambda}{\partial\rho}\gamma_i{}^j x^i{}_4 = 0$$

Thus, for the ideal compressible fluid (when $Q_i = Q_A = 0$), Equation (14) becomes

$$\rho DS = 0 \quad (16)$$

i.e. the entropy of each particle is conserved.

2) Isotropic elastic medium: $\Lambda = \Lambda\ (\gamma^{\circ pq},\ \gamma^{\wedge pq};\ S)$, where [1, 3 and 8] $\gamma^{\circ pq}$ is a tensor characterizing spatial distances in the initial state. By definition [3], $\partial\gamma^{\circ pq} / \partial\xi^4 = 0$.

Let us find the vector F^j

$$F^j = \frac{\partial\Lambda}{\partial\gamma^\wedge{}_{pq}}\frac{\partial\gamma^\wedge{}_{pq}}{\partial x^i{}_s}x^j{}_s x^i{}_4 = \frac{\partial\Lambda}{\partial\gamma^\wedge{}_{pq}}[(\gamma_{il}\gamma_n{}^j + \gamma_{in}\gamma_l{}^j)x^l{}_p x^n{}_q]\,x^i{}_4 = 0$$

We see, that (16) is also fulfilled in this case.

Let us now consider a model, the Lagrangian Λ of which depends on the arguments

$$g^\wedge{}_{pq},\quad u^\wedge{}_r,\quad \rho,\quad \nabla^\wedge{}_k\rho$$

Using Formulas given in [3] we obtain the derivatives of the above arguments with respect to x^i, and $\nabla_k x^i$. (*)

$$F^{\wedge j} = -2\frac{\partial\Lambda}{\partial g^{\wedge pq}}u^{\wedge p}g^{\wedge qj} + \left(u^{\wedge p}\ \frac{\partial\Lambda}{\partial u^{\wedge p}}\right)u^{\wedge j} - \frac{\partial\Lambda}{\partial\nabla^\wedge{}_j\rho}D\rho$$

If Λ depends only on the traces

$$D\rho = u^{\wedge k}\nabla^\wedge{}_k\rho, \qquad g^{\wedge pq}(\nabla^\wedge{}_p\rho)(\nabla^\wedge{}_q\rho)$$

then it is easy to see, that $F^j = 0$.

In the above cases the equality $DS = 0$ continues to hold also when the arguments of Λ include μ^A, $\nabla_j\mu^A$, and $\nabla_i\nabla_j\mu^A$.

The right-hand side of (14) is not always equal to zero. Suppose for example, that Λ depends on the vector $n^{\wedge k}$ which is not subject to variation, dependence being given by the trace $D_n\rho = n^{\wedge k}\nabla^\wedge{}_k\rho$. We assume for simplicity, that $\partial n^{\wedge k} / \partial\xi^4 = 0$, a ξ^4 we shall choose ξ^4 to be the length of arc of the world line. Then

*) Here we use the metric in the space-time manifold with the $(- - - +)$ sign convention.

$$\rho DS = \frac{1}{\Theta} \nabla^{\wedge}{}_k \left[\left(\frac{\partial \Lambda}{\partial D_n \rho} D\rho \right) n^{\wedge k} \right] \neq 0$$

In general, when a model determinable by the given Lagrangian Λ is constructed, we must, in accordance with the physical sense of the model, separate out the appearance of entropy σ and choose Λ so, as to satisfy the inequality $\sigma \geq 0$.

Model of viscous, heat conducting fluid. As an example, we shall formulate the variational principle for the model of viscous, heat conducting fluid in the general case of a nonsymmetric energy impulse tensor. Here the Lagrangian Λ depends not only on the density ρ and entropy S , but also on the antisymmetric tensor of angular velocity Ω_{ij} (by definition, $\Omega_{\alpha\beta}^*$ describe the mean angular velocity of particles in the intrinsic coordinate system, and $\Omega_{\alpha 4}^* = 0$) . We shall also introduce the tensor ω_{ij} through the relation $\Omega_{ij} = D\omega_{ij}$, with ω_{ij} taking the part played by μ^A in the general formalism, and subject it to variations. In particular, when

$$\Lambda = -\rho U(\rho, S) + \rho U_r, \; U_r = {}^1/_4 I^{ijkl} \Omega_{ij}\Omega_{kl},$$

where U_r is the macroscopic energy of the internal rotational motion of particles and I^{ijkl} is the moment of inertia tensor, we find, that the part of the energy impulse tensor defined by the Lagrangian Λ , has the form $P_{(\Lambda)}{}^{ij} = -p\gamma^{ij} + (\rho U + \rho U_r) u^i u^j$.

Functional δW^* shall be given in the form

$$\delta W^* = \int_V \left\{ \rho T \delta S - Q_i \delta x^i - \frac{1}{2} H^{ij} \delta\omega_{ij} \right\} d\tau - \int_\Sigma Q_i{}^j \delta x^i n_j d\sigma$$

Equations of motion (5) yield $\nabla_j P_i{}^j = Q_i + \nabla_j Q_i{}^j$. The requirement that the divergence of the energy impulse tensor brcomes zero, gives $Q_i = -\nabla_j Q_i{}^j$. Euler equations for ω_{ij} represent the equations of the balance of the internal angular momentum $\rho D M^{ij} + H^{ij} = 0$ (here $M^{ij} = 1/\rho\,(\partial\Lambda/\partial\Omega_{ij})$ is the tensor of internal angular momentum). Tensor H^{ij} arises from the lack of symmetry of the enegry impulse tensor and from the internal and mass moments. We shall assume the latter to be equal to zero and put $H^{ij} = P^{ji} - P^{ij}$.

Utilizing the obtained relationships, we can write the equation of entropy balance

$$\rho T D S = - u^i \nabla_j Q_i{}^j + {}^1/_2 H^{ij}\Omega_{ij} \tag{17}$$

Let us now make the choice of moving coordinate system more specific. We know [9], that in the investigation of irreversible processes velocity field and the corresponding moving coordinate system can be introduced in two different ways, namely, a system can be chosen wich is associated with the medium, or associated with the mass. We shall consider the moving system associated with the medium, hence the 4-velocity vector u^i entering (17) is the 4-velocity vector of the medium (*)

For any tensor of the 2nd rank and in particular for the tensor $Q_i{}^j$, we can write

$$Q^{ij} = s^{ij} + u^i I^j + G^i u^j + Q u^i u^j \tag{18}$$

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*) In the theory of relativity we can, generally speaking, consider two different 4-velocity vectors in describing the motion of continuous medium. These are the kinematic 4-velocity vector defining velocity of the medium and obtained by averaging the microscopic motion, and the dynamic 4-velocity vector which defines the mass velocity and is the intrinsic vector of the energy impulse tensor. Corresponding theories which use in constructing the energy impulse tensor either one or the other velocity field, diverge (in particular, determination of isotropy of continuous medium has different meanings). Expression for the symmetric energy impulse tensor of viscous fluid obtained using the concept of dynamic 4-velocity, is given in [10], while the symmetric energy impulse tensor of viscous heat conducting fluid in the system moving with the fluid, was investigated in [5]. Here we present a generalization to the case of an asymmetric energy impulse tensor.

$$s^{ij} = \gamma_k{}^i \gamma_l{}^j Q^{kl}, \quad I^j = \gamma_l{}^j u_k Q^{kl}, \quad G^i = \gamma_k{}^i u_l Q^{kl}, \quad Q = u_k u_l Q^{kl} \tag{19}$$

and, by (19), Equation (18) will be an identity. From (19) it follows, that

$$u_j s^{ij} = u_i s^{ij} = u_j I^j = u_i G^i = 0 \tag{20}$$

Putting (18) into (17) and taking (20) into account, we obtain

$$\rho DS = T^{-1} [s^{ij}(\nabla_j u_i - \Omega_{ij}) - \nabla_j I^j + G^i D u_i - \nabla_j (Q u^j)]$$

By the previous assumption, processes are reversible when $Q_i{}^j = 0$, consequently the appearance of entropy σ can be separated out as follows:

$$\rho DS = -\nabla_j (T^{-1} I^j) + \sigma$$

$$\sigma = I^j \nabla_j T^{-1} + T^{-1} (s^{ij} + G^i u^j)(\nabla_j u_i - \Omega_{ij}) - T^{-1} \nabla_j (Q u^j)$$

For the following discussion we shall assume that $Q = 0$.

We can consider the independent "thermodynamic flows" s^{ij}, I^j and G^i to be the functions of thermodynamic "forces" $\nabla_j T^{-1}$, $\nabla_j u_i$ and Ω_{ij}. The model of viscous, heat conducting fluid, is obtained under the assumption of linear realtionship existing between s^{ij}, I^j, G^i and $\nabla_j T^{-1}$, $\nabla_j u_i$, Ω_{ij}.

$$I^j = A^{jk} \nabla_k T^{-1} + B^{jkl}(\nabla_k u_l + \Omega_{kl}), \quad T^{-1}(s^{ij} + G^i u^j) = A^{ijk} \nabla_k T^{-1} + A^{ijkl}(\nabla_k u_l + \Omega_{kl}) \tag{21}$$

Vector G^i can be found from the last equation by contraction with u^j

$$T^{-1} G^i = u_j (A^{ijk} \nabla_k T^{-1} + A^{ijkl}(\nabla_k u_l - \Omega_{lk}))$$

Phenomenological coefficients A^{ij}, A^{ijk}, B^{jkl} and A^{ijkl} characterize properties of the medium and should possess its symmetry properties. Continuous medium may exhibit the symmetry of one of the crystallographic groups. General form of three-dimensional tensors invariant with respect to crystallographic groups, was obtained in [11]. Below, we shall consider the isotropic medium for which

$$A^{jk} = l_1 \gamma^{jk} + l_2 u^j u^k \quad A^{ijk} = k_1 \gamma^{ij} u^k + k_2 \gamma^{ik} u^j + k_3 \gamma^{jk} u^i + k_4 u^i u^j u^k$$

$$B^{jkl} = \mu_1 \gamma^{jl} u^k + \mu_2 \gamma^{jk} u^l + \mu_3 \gamma^{kl} u^j + \mu_4 u^j u^k u^l$$

$$A^{ijkl} = \nu_1 \gamma^{ij} \gamma^{kl} + \nu_2 \gamma^{ik} \gamma^{jl} + \nu_3 \gamma^{il} \gamma^{jk} + \nu_4 \gamma^{il} u^j u^k + \nu_5 \gamma^{ij} u^k u^l +$$

$$+ \nu_6 \gamma^{jl} u^i u^k + \nu_7 \gamma^{jk} u^i u^l + \nu_8 \gamma^{kl} u^i u^j + \nu_9 \gamma^{ik} u^j u^l + \nu_{10} u^i u^j u^k u^l \tag{22}$$

Quantities l_1, l_2, k_1,... represent scalar functions of the system of arguments (2) and their derivatives. Some of them become, by virtue of the relationships $u_i(s^{ij} + G^i u^j) = 0$ and $u_j I^j = 0$, equal to zero

$$l_2 = k_3 = k_4 = \mu_3 = \mu_4 = \nu_6 = \nu_7 = \nu_8 = \nu_{10} = 0$$

Coefficients μ_2, ν_5 and ν_9 will also not be significant, since the trace of $u^i \nabla_k u^i$ gives zero. Also it can be shown, that the inequality $\sigma \geq 0$ imposes the following restrictions on the coefficents $l_1, k_1, k_2, \mu_1, \nu_1, \nu_2, \nu_3$ and ν_4

$$k_1 = 0, \quad \nu_2 + \nu_3 \geqslant 0, \quad \nu_2 - \nu_3 \geqslant 0, \quad \nu_1 + (\nu_2 + \nu_3) / 3 \geqslant 0$$

$$\nu_4 \leqslant 0, \quad l_1 \leqslant 0, \quad (k_2 + \mu_1)^2 \leqslant 4 \nu_4 l_1$$

In conclusion, let us write the implicit expressions for the viscous stress tensor s^{ij}, for the vector of heat flow I^i and for the vector of angular momentum G^i

$$s^{ij} = \eta (\gamma^{ik} \nabla_k u^j + \gamma^{jk} \nabla_k u^i) + (\zeta - {}^2\!/_3 \eta) \gamma^{ij} \nabla_k u^k + \xi [\gamma^{ik}(\nabla_k u^j - \Omega^j{}_k) - \gamma^{jk}(\nabla_k u^i - \Omega^i{}_k)]$$

where

$$\eta \equiv (2T)^{-1} (\nu_2 + \nu_3) \geqslant 0, \quad \zeta \equiv T^{-1} [\nu_1 + (\nu_2 + \nu_3) / 3] \geqslant 0, \quad \xi \equiv (2T)^{-1} (\nu_2 - \nu_3) \geqslant 0$$

When $\xi = 0$, we obtain the symmetric viscous stress tensor. For vectors

I^i and G^i we have

$$\dot{I}^j = \varkappa\gamma^{jk}\nabla_k T + \mu_1 Du^j, \qquad \varkappa \equiv l_1 / T^2$$

$$G^i = \varkappa_1\gamma^{ik}\nabla_k T + T\nu_4 Du^i, \qquad \varkappa_1 \equiv k_2 / T$$

The equality $\mu_1 = \chi_1$ is the relativistic analog of Onsager relationships.

If the tensor Q^{ij} is symmetric, then $I^j = G^j$, hence $\chi = \chi_1$ and $\mu_1 = T\nu_4$. Then, for $\mu_1 = \chi_1$ we have

$$I^j = G^j = \varkappa(\gamma^{jk}\nabla_k T + Du^j)$$

which coincides with the known expression given in [5].

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